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LETTER TO THE EDITOR

Exact solution of an N -sublattice vertex model

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Abstract. The exact diagonalization of the diagonal-to-diagonal transfer matrix of an N -sublattice vertex model with interactions of the vertex-arrow type between sublattices is carried out using the generalized nested Bethe-ansatz method. An exact expression for the free energy is obtained involving the solution of an integral equation which is studied numerically.

Considerable progress in the construction and investigation of new exactly solvable models in two-dimensional statistical mechanics [1-4] has been achieved since the famous Baxter solution [5] of the eight-vertex model. In particular new lattice models in which the links may be in q different states (with $q > 2$) were solved using the nested Bethe-ansatz method [4, 6, 7]. A few two-sublattice models with interactions of the arrow-arrow type between sublattices were also solved using this method [8-10]. However, up to now, generalizations of these models for the case of an arbitrary number of interacting sublattices have not yet been constructed. Therefore the consideration of multi-sublattice models with another type of interaction between sublattices may be of interest.

In this letter we consider a vertex model of such a kind with N sublattices and with a new type of interaction between sublattices, namely vertex-arrow interactions. This model is a generalization of the two-sublattice vertex model studied in [11] and may be considered as the classical counterpart of a quantum model solved in [12].

The model may be formulated as follows. We consider a system of N zero-field six-vertex models as shown on figure 1 in the case $N = 3$. With each edge of the different

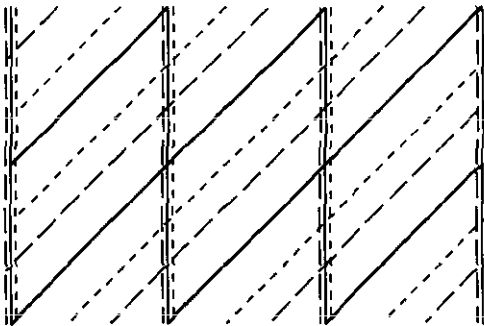


Figure 1. Geometry used to write the diagonal-to-diagonal transfer matrix of the N -sublattice model (with $N = 3$).

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sublattices is associated an arrow that points towards one of the two neighbouring vertices. At each vertex, we allow for the six standard arrow configurations obeying the ice rule (figure 2) with Boltzmann weights

$$\omega_1 = \omega_2 = a \quad \omega_3 = \omega_4 = b \quad \omega_5 = \omega_6 = c. \quad (1)$$

The interactions between sublattices are defined in the following way. From figure 1, it is clear that near each vertex of one sublattice one finds $N-1$ vertical arrows pertaining to the remaining sublattices. We suppose that the corresponding vertex weight depends on the direction of these arrows through

$$\{a, b, c\} = \{1, b e^{-s\eta}, \sqrt{1+b^2 e^{-2s\eta}}\} \quad (2)$$

where s is the number of nearest vertical arrows pointing down on other sublattices for the considered vertex ($s = 0, 1, 2, \dots, N-1$). So we have a system of interacting six-vertex models under the additional free-fermion condition $a^2 + b^2 - c^2 = 0$ within each sublattice. This is the price to pay for the introduction of interactions between sublattices while keeping an integrable model. These interactions between sublattices are infinite-ranged and their strength is parametrized by η ($\eta > 0$).

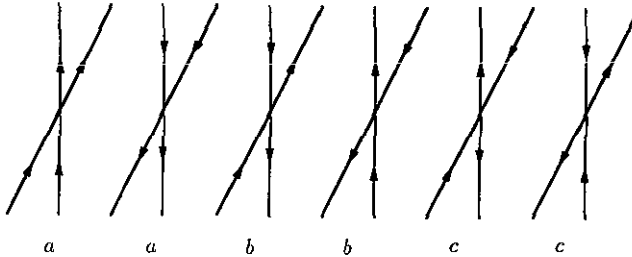


Figure 2. The six arrow configurations allowed at a vertex.

To calculate the partition function of the system, we use the diagonal-to-diagonal transfer matrix [8, 13]. In the thermodynamic limit, the free energy may be expressed in terms of the maximal eigenvalue of this matrix as

$$-\beta f = \lim_{L \rightarrow \infty} \frac{1}{L} \ln \Lambda_{\max} \quad (3)$$

where L is the number of vertices on a row in each sublattice and β is the inverse temperature. The transfer matrix \mathbf{T} relates the states of two successive rows of arrows on the lattice and the eigenvalue equation has the following form

$$\mathbf{T}\Psi = \Lambda\Psi. \quad (4)$$

Since on the transition from one row to the next, the number of down-pointing arrows in each sublattice is conserved, an obvious way to describe the state of a row is to specify the positions of the arrows directed downwards. Each cell of the system with coordinate x consists of N pairs of edges (x, τ) belonging to the different sublattices, where $\tau = 1$ and 2 for inclined and vertical edges, respectively. Let

$$\Psi_{\tau_1 \dots \tau_n}(x_1, \dots, x_{m_1} | x_{m_1+1}, \dots, x_{m_1+m_2} | \dots | x_{n-m_N+1}, \dots, x_n) \quad (5)$$

be the amplitude corresponding to the state of a row with down-pointing arrows on the edges $(x_1, \tau_1), \dots, (x_{m_1}, \tau_{m_1})$ of the first sublattice, on the edges $(x_{m_1+1}, \tau_{m_1+1}), \dots, (x_{m_1+m_2}, \tau_{m_1+m_2})$ of the second sublattice and so on. The total number of arrows directed downwards is then given by $n = \sum_{i=1}^N m_i$.

In order to write down the generalized Bethe ansatz [4, 6], we divide the domain of definition of Ψ in equation (5) into subdomains by means of the relation

$$(1, 1) \leq (x_{Q_1}, \tau_{Q_1}) \leq (x_{Q_2}, \tau_{Q_2}) \leq \dots \leq (x_{Q_n}, \tau_{Q_n}) \leq (L, 2) \quad (6)$$

where $[Q_1, \dots, Q_n]$ is a permutation over the numbers $1, 2, \dots, n$. The inequality $(x_i, \tau_i) < (x_j, \tau_j)$ means that either $x_i < x_j$ or $x_i = x_j, \tau_i = 1, \tau_j = 2$ or $x_i = x_j, \tau_i = \tau_j = 1$ ($i < j$). In the last case the arrows directed downwards are located on different sublattices. The equality $(x_i, \tau_i) = (x_j, \tau_j)$ is realized only when $x_i = x_j, \tau_i = \tau_j = 2$, i.e. with vertical arrows belonging to different sublattices. On figure 1 this means that (x_i, τ_i) is located on the left of (x_j, τ_j) . Within each of these subdomains, we write the amplitude in equation (5) in the form of the generalized Bethe ansatz [4, 6]

$$\Psi_{\tau_1 \dots \tau_n}(x_1, \dots, x_n) = \prod_l W_l^{n'_l} \sum_P (-1)^P (-1)^Q A_{P_1 \dots P_n}^{\alpha_1 \dots \alpha_n} \prod_{j=1}^n \psi_{\tau_{Q_j}}^{(k_{P_j})}(x_{Q_j}) \quad (7a)$$

$$W_l = \prod_{j=1}^{l-1} [(1 + b^2 e^{-2j\eta}) / (1 + b^2)]^{1/2}. \quad (7b)$$

The sum in (7a) is over all the permutations $P = [P_1, \dots, P_n]$ on $1, 2, \dots, n$. n'_l is the number of groups consisting of l vertical arrows directed downwards with the same coordinates. The $\psi_{\tau}^{(k)}(x)$'s are the amplitudes of one-particle states. Their explicit form is obtained by considering the case $n = 1$

$$\psi_1^{(k)}(x) = c^{-1}(\lambda - b) \exp(ikx) \quad \psi_2^{(k)}(x) = \exp(ikx) \quad (8a)$$

$$\lambda = \lambda(p) = e^{-ip} [b \cos p \pm \sqrt{1 + b^2 \cos^2 p}] \quad p = k/2 \quad (8b)$$

where λ is the eigenvalue of the one-particle problem. The eigenvalue corresponding to the amplitude (7a) is then

$$\Lambda = \prod_{j=1}^n \lambda(p_j). \quad (9)$$

The amplitude (7a) will satisfy the eigenvalue equation (4) by construction in the case where down-pointing arrows are situated at different lattice sites, i.e. within the subdomains (6). Equation (4) is also satisfied on the boundary of each subdomain provided the coefficients A_p^{α} satisfy the following equations

$$A_{P_1 \dots P_2 \dots}^{\alpha_1 \dots \alpha_2 \dots} = \sum_{\alpha, \beta=1}^N S_{\alpha_1 \alpha_2}^{\beta, \alpha} [\frac{1}{2}(M_{P_2} - M_{P_1})] A_{P_2 P_1 \dots}^{\alpha \beta \dots} \quad (10)$$

$$A_{P_1 \dots P_n}^{\alpha_1 \dots \alpha_n} = A_{P_2 \dots P_n P_1}^{\alpha_2 \dots \alpha_n \alpha_1} \exp(ik_{P_1} L)$$

where the non-vanishing elements of the S -matrix are

$$S_{\alpha\alpha}^{\alpha\alpha}(M) = 1 \quad S_{\alpha\beta}^{\alpha\beta}(M) = \sin M / \sin(M + i\eta)$$

$$S_{\alpha\beta}^{\beta\alpha}(M) = i \sinh \eta \exp[i \operatorname{sgn}(\beta - \alpha)M] / \sin(M + i\eta) \quad (11)$$

$$M_j = M(p_j) \quad M(p) = i \ln \lambda(p).$$

A necessary and sufficient condition for the compatibility of equation (10) is the fulfilment of the Yang-Baxter equations [1-4, 6]. In our case, the S -matrix which has a well known form [7, 14] satisfies these equations and we may use the quantum inverse

scattering method [2, 3, 15, 16] to solve (10). As a result, we obtain a system of transcendental equations for the p_j and additional unknown quantities $\Lambda_\gamma^{(k)}$

$$2Lp_j + \sum_{\alpha=1}^{\bar{m}_{N-1}} \Theta(M_j - \Lambda_\alpha^{(1)}; \eta') = 2\pi I_j \quad (j=1, 2, \dots, n)$$

$$\sum_{\sigma=\pm 1} \sum_{\alpha=1}^{\bar{m}_{N-k-\sigma}} \Theta(\Lambda_\gamma^{(k)} - \Lambda_\alpha^{(k+\sigma)}; \eta') - \sum_{\gamma'=1}^{\bar{m}_{N-k}} \Theta(\Lambda_\gamma^{(k)} - \Lambda_{\gamma'}^{(k)}; 2\eta') = 2\pi J_\gamma^{(N-k)} \quad (12)$$

$$1 \leq \gamma \leq \bar{m}_{N-k} \quad 1 \leq k \leq N-1 \quad (\Lambda_j^{(0)} \equiv M_j)$$

where

$$\Theta(M; \eta) = 2 \tan^{-1}(\coth \eta \tan \frac{1}{2}M) \quad -\pi \leq \Theta(M; \eta) < \pi$$

and I_j and $J_\gamma^{(k)}$ are integer (half-integer) numbers for odd (even) $\bar{m}_{N-1} + 1$ and $\bar{m}_{k-1} + m_{k+1}$ respectively,

$$\bar{m}_k = \sum_{j=1}^k m_j \quad (13)$$

is the total number of arrows directed downwards on the first, second, ... and k th sublattices.

The largest eigenvalue of the transfer matrix corresponds to the following values of I_j and $J_\beta^{(k)}$

$$I_{j+1} - I_j = 1 \quad J_{\beta+1}^{(k)} - J_\beta^{(k)} = 1 \quad (14)$$

therefore in the thermodynamic limit ($L \rightarrow \infty$), for fixed ratios n/L and m_k/L , we may assume that the values of p_j and $\Lambda_\gamma^{(k)}$ fill the intervals $[-Q, Q]$ and $[-B_k, B_k]$ uniformly with the densities $\rho(p)$ and $\sigma^{(k)}(\Lambda)$, respectively. Then, instead of (12), we obtain the following system of integral equations

$$2\pi\rho(p) = 1 + M'(p) \int_{-B_1}^{B_1} \Theta'(M(p) - \Lambda; \eta') \sigma^{(1)}(\Lambda) d\Lambda$$

$$2\pi\sigma^{(k)}(\Lambda) + \int_{-B_k}^{B_k} \Theta'(\Lambda - \Lambda'; 2\eta') \sigma^{(k)}(\Lambda') d\Lambda'$$

$$= \sum_{\sigma=\pm 1} \int_{-B_{k+\sigma}}^{B_{k+\sigma}} \Theta'(\Lambda - \Lambda'; \eta') \sigma^{(k+\sigma)}(\Lambda') d\Lambda' \quad (15)$$

$$\int_{-Q}^Q \rho(p) dp = \frac{n}{2L} \quad \int_{-B_k}^{B_k} \sigma^{(k)}(\Lambda) d\Lambda = \frac{\bar{m}_{N-k}}{L}$$

$$1 \leq k \leq N-1 \quad B_0 = Q \quad B_N = 0 \quad \sigma^{(0)}(\Lambda) \equiv \rho(\Lambda).$$

From the symmetry of the system, it is clear that Λ_{\max} corresponds to symmetrical configurations with the same number of arrows directed downwards on all sublattices. Then all $B_k = \pi$ and the Fourier transformation may be used to reduce the system of equations (15) to a single integral equation for the unknown function $\rho(p)$

$$2\pi\rho(p) - M'(p) \int_{-Q}^Q \varphi[M(p) - M(p')] \rho(p') dp' = 1 \quad (16)$$

where

$$\varphi(M) = 1 - \frac{1}{N} + 2 \sum_{n=1}^{\infty} \exp(-n\eta) \frac{\sinh[n\eta(N-1)]}{\sinh(n\eta N)} \cos(nM). \quad (17)$$

The free energy given by equation (3) then takes the following form

$$f = -2\beta^{-1} \int_{-Q}^Q \ln[b \cos p + \sqrt{1 + b^2 \cos^2 p}] \rho(p) dp. \quad (18)$$

The parameter Q in equations (16)–(18) must be chosen to minimize the right-hand side of equation (18).

Once Q has been determined, we may calculate the density of arrows pointing down

$$\rho = \frac{1}{N} \int_{-Q}^Q \rho(p) dp \quad (19)$$

from which the magnetization $y = 1 - 2\rho$ follows. We have carried out this program numerically in the quantum limit ($b \rightarrow 0$). The first term in the expansion of (18) then corresponds to the ground-state energy of the quantum model considered in [12]. The results of these calculations illustrate the dependence of the density ρ on the interaction parameter η (figure 3) and the number of sublattices N (figure 4) for arbitrary small values of b . From these figures it is clear that the model under consideration has a

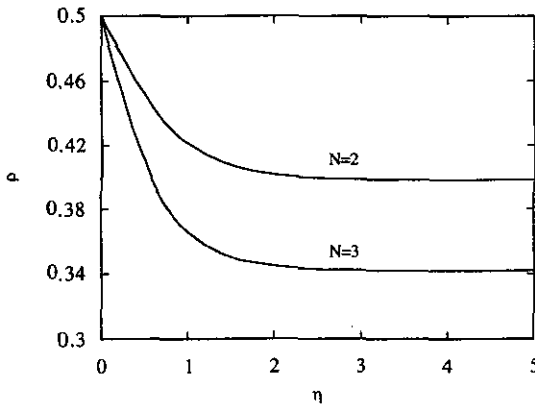


Figure 3. Density of arrows directed downwards as a function of the interaction parameter η .

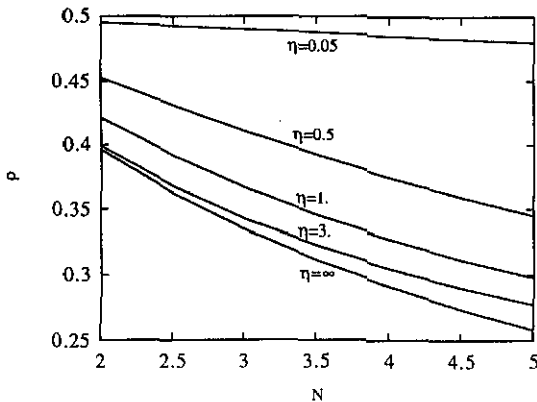


Figure 4. Density of arrows directed downwards as a function of the number of sublattices N .

finite zero-field magnetization. The reason for it is the asymmetry of the model under arrow reversal.

Thus we have obtained an exact expression for the free energy of the N -sublattice vertex model with interactions between sublattices of the vertex-arrow type. This solution is analytic with respect to b and η in the entire range of variation of the parameters except for the point $\eta=0$. This is due to the fact that this solution corresponds to the critical point of a more general model. The study of the critical behaviour of the model considered and its connections with other integrable systems will be the subject of a following investigation.

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References

- [1] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
- [2] Faddeev L D 1984 *Recent Advances in Field Theory and Statistical Mechanics (Les Houches Summer School Proc. Session XXXIX)* ed J-B Zuber and R Stora (Amsterdam: North-Holland) p 561
- [3] Kulish P P and Sklyanin E K 1982 *Lecture Notes in Physics* **151** 61
- [4] Gaudin M 1983 *La Fonction d'Onde de Bethe* (Paris: Masson)
- [5] Baxter R J 1971 *Phys. Rev. Lett.* **26** 832; 1972 *Ann. Phys., NY* **70** 197
- [6] Yang C N 1968 *Phys. Rev. Lett.* **19** 1312
- [7] Babelon O, de Vega H J and Viallet C M 1981 *Nucl. Phys. B* **190** [FS3] 542; 1982 *Nucl. Phys. B* **200** [FS4] 266
- [8] Bariev R Z 1982 *Theor. Math. Phys.* **49** 1021
- [9] Shastry B S 1986 *Phys. Rev. Lett.* **56** 1529; 1986 *Phys. Rev. Lett.* **56** 2453; 1988 *J. Stat. Phys.* **50** 57
- [10] Bariev R Z 1990 *Theor. Math. Phys.* **82** 218
- [11] Bariev R Z and Truong T T 1992 submitted for publication
- [12] Bariev R Z 1991 *J. Phys. A: Math. Gen.* **24** L919
- [13] Truong T T and Schotte K D 1983 *Nucl. Phys. B* **220** [FS8] 77
- [14] Chudnovsky D V and Chudnovsky G V 1980 *Phys. Lett.* **79A** 36
- [15] Takhtadzhyan L A and Faddeev L D 1979 *Russ. Math. Surveys* **34** 11
- [16] Belavin A A 1979 *Phys. Lett.* **87B** 193